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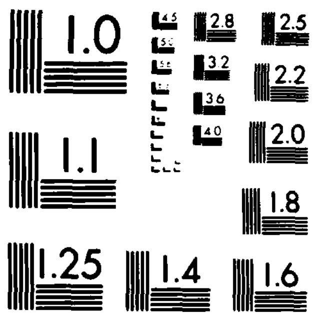
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MRC Technical Summary Report # 2530

A LOCAL EXISTENCE AND UNIQUENESS THEOREM FOR A K-BKZ-FLUID

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June 1983

(Received April 6, 1983)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

We consider the three-dimensional motion of a viscoelastic liquid occupying all space. The constitutive law is assumed to be of the form suggested by Kaye [13] and Bernstein, Kearsley and Zapas [2]. An existence and uniqueness result for solutions of the initial value problem on sufficiently short time intervals is proved using Kato's theory of quasi-linear hyperbolic equations.

AMS (MOS) Subjection Classifications: 35L15, 45K05, 47D05, 76A10

Key Words: Quasilinear Hyperbolic Equations, Viscoelastic Liquids, Semigroup theory

Work Unit Number 1 - Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062, Mod 2.

SIGNIFICANCE AND EXPLANATION

The existence theory for models of viscoelastic fluids has so far not been very well developed, in particular in three dimensional situations. Here, ~~we~~ the author proves an existence theorem for a particular class of models, suggested by Kaye and Bernstein, Kearsley and Zapas. This theory is based on a postulated analogy with hyperelasticity. It is assumed that the fluid occupies all of space. Abstract methods developed originally for quasilinear hyperbolic systems can be used to prove the well-posedness of the initial value problem.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A LOCAL EXISTENCE AND UNIQUENESS THEORM
FOR A K-BKZ-FLUID

Michael Renardy

1. Introduction

The purpose of this paper is to show the well-posedness of the initial history value problem for a class of viscoelastic materials with memory. The constitutive law is assume to be of the form proposed by Kaye [13] and Bernstein, Kearsley and Zapas [2]. Their theory is motivated by a formal analogy with classical hyperelasticity [21]. The equilibrium configuration is replaced by a configuration previously occupied by the material. It is assumed that the stress contributions from all the previous configurations superpose in an additive fashion, leading mathematically to a convolution integral.

We consider a material occupying all of $\mathbb{R}^3$, which is assumed incompressible. By $\underline{\zeta} = (\zeta^1,\zeta^2,\zeta^3)$ we denote body coordinates, while $\underline{y}(\underline{\zeta},t) = (y^1,y^2,y^3)$ denotes the position of material points in space. The equation of motion is given by

$$\rho \ddot{y}^i = \frac{\partial}{\partial\zeta^s}\left(\frac{\partial y^i}{\partial\zeta^r}\pi^{rs}\right) + g^i(\underline{\zeta},t)$$

(1.1)

$$\det\left(\frac{\partial y^i}{\partial\zeta^j}\right) = 1 \quad .$$

Here $\underline{\underline{\pi}}$ denotes the convected stress tensor, and the g^i are the components of the body force (we could allow g^i to depend on $\underline{y}$, but omit this for simplicity). The Einstein convention is used throughout this paper.

We also need the following notations:

$$\underline{\underline{\gamma}} = (\gamma_{ij}) = \left(\frac{\partial y^r}{\partial\zeta^i}\frac{\partial y^r}{\partial\zeta^j}\right) \quad , \quad \gamma^{ij}\gamma_{jk} = \delta^i_k$$

$$I_1(\underline{\zeta},t,\tau) = \mathrm{tr}(\underline{\underline{\gamma}}(\underline{\zeta},t)\underline{\underline{\gamma}}^{-1}(\underline{\zeta},\tau))$$

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$$I_2(\underline{\xi},t,\tau) = \mathrm{tr}(\underline{\gamma}^{-1}(\underline{\xi},t)\underline{\gamma}(\underline{\xi},\tau)) \quad .$$

The K-BKZ constitutive model is described in terms of a function W, which is the analogue of the strain energy in classical hyperelasticity. W is a real-valued function of the two arguments I_1 and I_2. The constitutive law of K-BKZ is

$$\pi^{rs} = -p\gamma^{rs} + \int_{-\infty}^{t} a(t-\tau)\,[\frac{\partial W}{\partial I_1} \cdot \gamma^{rs}(\tau) - \frac{\partial W}{\partial I_2}\,\gamma^{rp}(t)\gamma_{pq}(\tau)\gamma^{qs}(t)]d\tau \quad . \tag{1.2}$$

Here p denotes the undetermined pressure. The kernel a is positive and monotone decreasing. We also assume that W and a are smooth functions and that a decays sufficiently fast (e.g., exponentially) at infinity. In particular, we do not admit that a is singular at 0. Such singularities, however, are physically possible and are in fact predicted by some molecular theories, e.g. [5]. Thus the requirement that a is smooth must be regarded as a restriction on the material.

The K-BKZ theory includes various models suggested in the literature - some of them based on molecular theories - as special cases [3], [5], [7], [15-17], [22], [23]. A crucial hypothesis needed for well-posedness of the initial value problem is that the equation of motion is "hyperbolic", or, equivalently, that the corresponding quasi-static equation is "elliptic". This requirement leads to an inequality involving first and second derivatives of W. In chapter 2, we give a sufficient condition for ellipticity. We show that ellipticity holds if W is monotone in each argument and is a convex function of $\sqrt{I_1}$ and $\sqrt{I_2}$. These conditions imply in particular that, in simple shear flow, N_1 is positive, N_2 is negative and $|N_1| > |N_2|$. These consequences are consistent with experimental results on polymer melts and solutions.

Our well-posedness theorem is based on Kato's theory of quasilinear hyperbolic systems [8], [10-12]. The case of classical hyperelasticity is considered in [8], and, to some extent, our arguments will be analogous. However, the history dependence and incompressibility conditions introduce complicating features. As a result, Kato's assumptions cannot be verified immediately, using the equations as they stand, but only after going through some transformations and substitutions.

We shall not consider (1.1) itself, but its second time-derivative. Equation (1.1) and its first time derivative are then used to express $\underset{\sim}{y}$ and $\dot{\underset{\sim}{y}}$ in terms of $\ddot{\underset{\sim}{y}}$ and $\dddot{\underset{\sim}{y}}$. This elimination is discussed in chapter 3. Because of ellipticity, there is a gain of regularity associated with it. Roughly speaking, this serves to improve the smoothness of the "coefficients" in a quasilinear formulation of the problem.

In chapter 4, we finally pose the evolution problem in a form accessible to Kato's method. The delay equation is reformulated as an abstract evolution problem on a history space. In contrast with some previous approaches to similar problems, however, this history space only involves the history back to the initial time $t = 0$ rather than all the way back to $t = -\infty$. In the previous literature, existence theorems for "hyperbolic" models of viscoelastic fluids mostly concern linearized [9], [20] or nonlinear, but one-dimensional problems [4], [18]. The present problem has been studied by Kim [14] in the special case $W = I_2$. Kim's method is based on artificial viscosity and differs substantially from the approach used here.

2. A sufficient condition for ellipticity

For abbreviation, let $\tilde{\underline{F}}$ be defined by $\tilde{F}^i_j = \frac{\partial y^i}{\partial \zeta^j}$, and let $\underline{F}(t,\tau) = \tilde{\underline{F}}(t)\tilde{\underline{F}}^{-1}(\tau) = \frac{\partial \underline{y}(t)}{\partial \underline{y}(\tau)}$. Then $I_1 = \mathrm{tr}(\underline{F}^T\underline{F})$, and $I_2 = \mathrm{tr}(\underline{F}^{-1}\underline{F}^{-T})$, so that W is a function of $\underline{F}$. The right hand side of (1.1) can be rewritten as follows

$$\frac{\partial}{\partial \zeta^s}\left(\frac{\partial y^i}{\partial \zeta^r}\pi^{rs}\right) = -\frac{\partial p}{\partial \zeta^s}\frac{\partial \zeta^s}{\partial y^i} + \frac{1}{2}\int_{-\infty}^{t} a(t-\tau)\cdot$$

$$\frac{\partial^2 W}{\partial F^i_p \partial F^j_r}\left[\frac{\partial^2 y^j}{\partial \zeta^q \partial \zeta^s}\frac{\partial \zeta^q}{\partial y^r(\tau)}\frac{\partial \zeta^s}{\partial y^p(\tau)} + \frac{\partial y^j}{\partial \zeta^q}\right. \tag{2.1}$$

$$\left.\frac{\partial}{\partial \zeta^s}\left(\frac{\partial \zeta^q}{\partial y^r(\tau)}\right)\frac{\partial \zeta^s}{\partial y^p(\tau)}\right]d\tau \quad .$$

Let $\bar{F}^p_i = \frac{\partial y^p(\tau)}{\partial y^i(t)}$ denote the entries of $\underline{F}^{-1}$. The right hand side of (2.1) remains unchanged if $\frac{\partial^2 W}{\partial F^i_p \partial F^j_r}$ is replaced by $\frac{\partial^2 W}{\partial F^i_p \partial F^j_r} + K\bar{F}^p_i\bar{F}^r_j$. The additional term vanishes as a result of the incompressibility condition. In the subsequent chapters, we shall assume that the following strong ellipticity condition is satisfied:

(E). If $K > 0$ is chosen large enough, then

$$\left(\frac{\partial^2 W}{\partial F^i_p \partial F^j_r} + K\bar{F}^p_i\,\bar{F}^r_j\right)\lambda^i\lambda^j\mu_p\mu_r \geq C|\lambda|^2|\mu|^2, \quad C > 0 \quad .$$

In terms of the original arguments of W, I_1 and I_2, this condition has a rather indirect form. However, we have the following sufficient criterion, which is easily expressed in terms of I_1 and I_2.

Lemma 2.1:

If W is monotone in each argument, strictly monotone in I_1 or I_2, and a convex function of $\sqrt{I_1}$ and $\sqrt{I_2}$, then (E) holds.

Proof:

We have

$$\frac{\partial^2 W}{\partial F^i_p \partial F^j_r} = 2\,\frac{\partial W}{\partial I_1}\,\delta_{ij}\delta^{pr}$$

$$+ 2\,\frac{\partial W}{\partial I_2}\,(\overline{F}^p_j\,\overline{F}^r_i\,\overline{F}^p_\sigma\,\overline{F}^p_\sigma$$

$$+ \overline{F}^p_i\,\overline{F}^p_j\,\overline{F}^r_\sigma\,\overline{F}^p_\sigma$$

$$+ \overline{F}^p_i\,\overline{F}^p_\sigma\,\overline{F}^p_j\,\overline{F}^r_\sigma)$$

$$+ 4\,\frac{\partial^2 W}{\partial I_1^2}\,F^i_p F^j_r$$

$$- 4\,\frac{\partial^2 W}{\partial I_1 \partial I_2}\,(F^i_p\,\overline{F}^p_j\,\overline{F}^r_\sigma\,\overline{F}^p_\sigma + F^j_r\,\overline{F}^p_i\,\overline{F}^p_\sigma\,\overline{F}^p_\sigma)$$

$$+ 4\,\frac{\partial^2 W}{\partial I_2^2}\,(\overline{F}^p_i\,\overline{F}^p_\sigma\,\overline{F}^p_\sigma\,\overline{F}^\mu_j\,\overline{F}^r_\nu\,\overline{F}^\mu_\nu) \quad .$$

Moreover, $\dfrac{\partial W}{\partial\sqrt{I_1}} = 2\sqrt{I_1}\,\dfrac{\partial W}{\partial I_1}$

$$\frac{\partial^2 W}{(\partial\sqrt{I_1})^2} = 2\,\frac{\partial W}{\partial I_1} + 4\,I_1\,\frac{\partial^2 W}{\partial I_1^2} \quad .$$

Using this, we obtain

$$\frac{\partial^2 W}{\partial F^i_p \partial F^j_r} = \frac{\partial W}{\partial\sqrt{I_1}}\,[\frac{1}{\sqrt{I_1}}\,\delta_{ij}\delta^{pr} - \frac{1}{I_1^{3/2}}\,F^i_p F^j_r]$$

$$+ \frac{\partial W}{\partial\sqrt{I_2}}\cdot\,[\frac{1}{\sqrt{I_2}}\,(\overline{F}^p_j\,\overline{F}^r_i\,\overline{F}^p_\sigma\,\overline{F}^p_\sigma + \overline{F}^p_i\,\overline{F}^p_j\,\overline{F}^r_\sigma\,\overline{F}^p_\sigma$$

$$+ \overline{F}^p_i\,\overline{F}^p_\sigma\,\overline{F}^p_j\,\overline{F}^r_\sigma)$$

$$- \frac{1}{I_2^{3/2}}\,(\overline{F}^p_i\,\overline{F}^p_\sigma\,\overline{F}^p_\sigma\,\overline{F}^\mu_j\,\overline{F}^r_\nu\,\overline{F}^\mu_\nu)]$$

$$+ \frac{1}{I_1}\,\frac{\partial^2 W}{(\partial\sqrt{I_1})^2}\,F^i_p F^j_r$$

$$- \frac{1}{\sqrt{I_1 I_2}} \frac{\partial^2 W}{\partial\sqrt{I_1}\ \partial\sqrt{I_2}} (F^i_p\, \overline{F}^\rho_j\, \overline{F}^r_\sigma\, \overline{F}^\rho_\sigma + F^j_r\, \overline{F}^\rho_i\, \overline{F}^p_\sigma\, \overline{F}^\rho_\sigma)$$

$$+ \frac{1}{I_2} \frac{\partial^2 W}{(\partial\sqrt{I_2})^2} (\overline{F}^\rho_i\, \overline{F}^p_\sigma\, \overline{F}^\rho_\sigma\, \overline{F}^\mu_j\, \overline{F}^r_\nu\, \overline{F}^\mu_\nu) \quad .$$

The first term leads to $\frac{\partial W}{\partial\sqrt{I_1}} \frac{1}{\sqrt{I_1}} [|\lambda|^2|\mu|^2 - \frac{1}{I_1} (\underline{\mu}\cdot\underline{\underline{F}}\ \underline{\lambda})^2] > 0$. The last three terms yield a positive contribution, if W is convex. To estimate the second term, let us put $\overline{F}^\rho_i\lambda^i = \tilde{\lambda}^\rho$, $F^p_\sigma F^\rho_\sigma = \alpha^{p\rho}$. We find

$$\frac{\partial W}{\partial\sqrt{I_2}} \cdot \frac{1}{\sqrt{I_2}} (\tilde{\lambda}^\rho\, \tilde{\lambda}^r\, \alpha^{p\rho}\mu_p\mu_r + \tilde{\lambda}^\rho\, \tilde{\lambda}^p\, \alpha^{\rho r}\mu_p\mu_r$$

$$+ \tilde{\lambda}^\rho\, \tilde{\lambda}^\rho\, \alpha^{pr}\mu_p\mu_r - \frac{1}{I_2}\, \tilde{\lambda}^\rho\, \tilde{\lambda}^\mu\, \alpha^{p\rho}\alpha^{r\mu}\mu_p\mu_r)$$

$$= \frac{\partial W}{\partial\sqrt{I_2}} \cdot \frac{1}{\sqrt{I_2}} [2(\underline{\tilde{\lambda}}\cdot\underline{\mu})(\underline{\tilde{\lambda}}\cdot\underline{\underline{\alpha}}\cdot\underline{\mu}) + |\underline{\tilde{\lambda}}|^2(\underline{\mu}\cdot\underline{\underline{\alpha}}\cdot\underline{\mu}) - \frac{1}{I_2} (\underline{\tilde{\lambda}}\cdot\underline{\underline{\alpha}}\cdot\underline{\mu})^2] \quad .$$

The term $K\overline{F}^p_i\, \overline{F}^r_j$ gives a a contribution $K\cdot(\tilde{\lambda}\cdot\mu)^2$. Since $I_2 = \mathrm{tr}\,\alpha$, we have $|\tilde{\lambda}|^2(\underline{\mu}\cdot\underline{\underline{\alpha}}\cdot\underline{\mu}) - \frac{1}{I_2} (\underline{\tilde{\lambda}}\cdot\underline{\underline{\alpha}}\cdot\underline{\mu})^2 > 0$, and the whole term can be made positive by chosing K large enough. This proves the lemma.

3. Solution of a nonlinear elliptic Volterra equation

Throughout the remainder of the paper, it is assumed that (E) holds with a suitable constant $K \in \mathbb{R}$. For abbreviation, we write $A_{ij}^{pr} = \frac{\partial^2 W}{\partial F_p^i F_r^j} + K \overline{F}_i^p \overline{F}_j^r$. Thus A_{ij}^{pr} depends on the gradients of $y(t)$ and $y(\tau)$, and it is symmetric in (i,p) and (j,r). Equation (1.1) is rewritten in the following equivalent form

$$h^i := \rho \ddot{y}^i - \lambda(y^i - \zeta^i) - g^i =$$

$$- \frac{\partial p}{\partial \zeta^s} \frac{\partial \zeta^s}{\partial y^i} + \frac{1}{2} \int_{-\infty}^{t} a(t-\tau) A_{ij}^{pr} \cdot \frac{\partial}{\partial \zeta^s} \tag{3.1}$$

$$\left[\frac{\partial y^j}{\partial \zeta^q} \frac{\partial \zeta^q}{\partial y^r(\tau)} \frac{\partial \zeta^s}{\partial y^p(\tau)}\right] d\tau - \lambda(y^i - \zeta^i)$$

$$0 = \det\left(\frac{\partial y^i}{\partial \zeta^j}\right) - 1 \quad .$$

It is convenient to split the history occurring in the integral into a known and an unknown part. Let the initial time be $t = 0$. Then $\underline{y}(\tau)$, $\tau < 0$ is considered known, and we shall denote it by $\underline{y}_0(\tau)$. On the interval $[0,t]$, we perform a rescaling: $\hat{\underline{y}}(\sigma) = \underline{y}(\sigma t)$, so that σ ranges from 0 to 1. Equation (3.1) can be rewritten as an equation for $\hat{\underline{y}}$ as follows:

$$\hat{h}^i(\sigma) = - \frac{\partial \hat{p}(\sigma)}{\partial \zeta^s} \frac{\partial \zeta^s}{\partial \hat{y}^i(\sigma)} + \frac{1}{2} \int_{-\infty}^{0} a(\sigma t - \tau) \cdot$$

$$A_{ij}^{pr}(\hat{y}(\sigma), y_0(\tau)) \frac{\partial}{\partial \zeta^s} \left[\frac{\partial \hat{y}^j(\sigma)}{\partial \zeta^q} \frac{\partial \zeta^q}{\partial y_0^r(\tau)} \frac{\partial \zeta^s}{\partial y_0^p(\tau)}\right] d\tau$$

$$+ \frac{1}{2} t \int_0^{\sigma} a(\sigma t - \sigma' t) A_{ij}^{pr}(\hat{y}(\sigma), \hat{y}(\sigma')) \cdot \frac{\partial}{\partial \zeta^s} \tag{3.2}$$

$$\left[\frac{\partial \hat{y}^j(\sigma)}{\partial \zeta^q} \frac{\partial \zeta^q}{\partial \hat{y}^r(\sigma')} \frac{\partial \zeta^s}{\partial \hat{y}^p(\sigma')}\right] d\sigma' - \lambda(\hat{y}^i(\sigma) - \zeta^i)$$

$$0 = \det\left(\frac{\partial \hat{y}^i(\sigma)}{\partial \zeta^i}\right) - 1 \quad .$$

The term involving the unknown part of the history (from 0 to t) is now multiplied by a factor of t, Hence it can be treated as a perturbation for t small enough. If we set $t = 0$, then the right hand side of (3.2) acts pointwise with respect to σ and, for each $\sigma \in [0,1]$, we have a nonlinear elliptic system for $\hat{\underline{y}}(\sigma)$ and $\hat{p}(\sigma)$.

We assume that we have a solution $\hat{\underline{y}}(\sigma) \equiv \underline{y}(0)$, $\hat{p}(\sigma) \equiv p(0)$ for $t = 0$ given by the initial conditions of the problem. If the linearization of the right hand side of (3.2) at this given solution is invertible, then the implicit function theorem can be used to show the existence of solutions to (3.2) for small t.

More precisely, let n be any integer > 1. We assume that the initial history is such that $\underline{y}_0(\tau) - \underline{\zeta}$ is continuous and bounded from $(-\infty,0]$ into $H^{n+2}(\mathbb{R}^3)$ and that it satisfies the incompressibility condition. Let the initial values $\underline{h}(0) \in H^n$, $\underline{y}(0) \in \underline{\zeta} + H^{n+2}$ and $\nabla p(0) \in H^n$ satisfy (3.1) for $t = 0$. Then the constant functions $\hat{\underline{h}}(\sigma) \equiv \underline{h}(0)$ etc. satisfy (3.2) for $t = 0$. Condition (E) guarantees that the linearization of the right hand side of (3.2) (for $t = 0$) is an elliptic system in the sense of Agmon, Douglis and Nirenberg [1], [6], moreover, it is invertible if $\lambda > 0$ is chosen large enough. The existence of a weak solution is established by variational methods using Gårding's inequality. Regularity can then be obtained using the standard trick of estimating difference quotients. According to the implicit function theorem, equation (3.2) can be solved uniquely for $\hat{\underline{y}} \in C([0,1]; \underline{\zeta} + H^{n+2})$, $\nabla\hat{p} \in C([0,1]; H^n)$ as functions of $\hat{\underline{h}} \in C([0,1]; H^n)$, provided t is small enough. For technical reasons, we shall have to use a different topology for the σ-dependence in chapter 4, but a similar argument will apply also with that topology.

By applying the same procedure to the time differentiated version of (1.1), we can resolve for $\underline{\dot{y}}$ and $\dot{p}$ as functions of $\underline{\dot{h}}$, $\hat{\underline{y}}$ and $\hat{p}$.

Remark: The reader should be careful to distinguish between $\underline{\dot{y}}$ and $\dot{\hat{\underline{y}}}$. We have $\underline{\dot{y}}(\sigma) = \underline{\dot{y}}(\sigma t)$, but $\dot{\hat{\underline{y}}}(\sigma) = \frac{d}{dt}\,\underline{y}(\sigma t) = \sigma\underline{\dot{y}}(\sigma t)$.

4. The hyperbolic Cauchy problem

We use the abbreviations: $\underline{u} = \dot{\underline{y}}$, $\underline{v} = \ddot{\underline{y}} - \frac{\lambda}{\rho}(\underline{y} - \underline{\zeta})$, $\underline{w} = \dddot{\underline{y}} - \frac{\lambda}{\rho}\dot{\underline{y}}$, $q = \dot{p}$, $\phi = \ddot{p}$, where λ is as in the previous chapter. The second time derivative of equation (1.1) can then be written as follows:

$$\dot{\underline{v}} = \underline{w}$$

$$\rho \dot{w}^i = -\frac{\partial \phi}{\partial \zeta^s}\frac{\partial \zeta^s}{\partial y^i} + \frac{1}{2}\int_{-\infty}^{t} a(t-\tau)A^{pr}_{ij}\frac{\partial}{\partial \zeta^s} \tag{4.1}$$

$$\left\{\frac{\partial v^j}{\partial \zeta^q}\frac{\partial \zeta^q}{\partial y^r(\tau)}\frac{\partial \zeta^s}{\partial y^p(\tau)}\right\}d\tau + \ddot{g}^i$$

$$+ \Phi^i\{\underline{y},\underline{u},\underline{v},p,q\}$$

$$\frac{\partial w^i}{\partial \zeta^j}\frac{\partial \zeta^j}{\partial y^i} = \Psi\ \{\underline{y},\underline{u},\underline{v}\} \quad .$$

Here $\underline{\Phi}$ is an expression involving spatial derivatives of $\underline{y}$ and $\underline{u}$ up to the second order, and spatial derivatives of $\underline{v}$, p and q up to the first order. Ψ depends on first spatial derivatives of $\underline{y}$, $\underline{u}$ and $\underline{v}$. Thus the $\underline{\Phi}$ - and Ψ - terms represent perturbations of lower differential order than the other terms in the equation. Of course, we think of $\underline{y}$, $\underline{u}$, p and q as being expressed in terms of $\underline{v}$ and $\underline{w}$ as described in section 3.

In order to represent (4.1) as an evolution problem, we have to eliminate the variable ϕ. As in Navier-Stokes theory, this is done by using the Hodge projection. The difference is that, since we work in Lagrangian coordinates, the Hodge projection is a function of the displacement $\underline{y}$. Let $\underline{y} \in \underline{\zeta} + H^{n+1}$ $(n > 2)$ be such that its gradient satisfies the incompressibility condition. We define an orthogonal projection $P(\underline{y})$ in $L^2(\mathbb{R}^3)$ to be such that its range consists of all vectorfields $\underline{u} \in L^2$ which satisfy $\frac{\partial u^i}{\partial \zeta^j}\frac{\partial \zeta^j}{\partial y^i} = 0$, while its nullspace contains all vectorfields of the form $\frac{\partial \zeta^s}{\partial y^i}\frac{\partial q}{\partial \zeta^s}$, where $\nabla q \in L^2$. In other words, $P(\underline{y})$ is the image of the Hodge projection under the coordinate transformation $\underline{y} \to \underline{\zeta}$. The projection $P(\underline{y})$ is continuous from H^j into itself for $j \leq n$, and it can be shown [19] that $P(\underline{y})$ - as an operator in H^j - depends smoothly on $\underline{y} \in \underline{\zeta} + H^{n+1}$.

Obviously, ϕ is eliminated from (4.1) by applying the projection $P(\underline{y})$. Moreover, the incompressibility condition requires that

$$\frac{\partial v^i}{\partial \zeta^s} \frac{\partial \zeta^s}{\partial y^i} = \frac{\partial u^i}{\partial \zeta^s} \frac{\partial \zeta^s}{\partial y^k} \frac{\partial u^k}{\partial \zeta^\ell} \frac{\partial \zeta^\ell}{\partial y^i} .$$

This can be used to express $(1 - P(\underline{y}))\underline{v}$ in terms of $\underline{y}$ and $\underline{u}$. More precisely, $(1 - P(\underline{y}))\underline{v} \in H^{n+2}$ is a smooth function of $\underline{u} \in H^{n+2}$, $\underline{y} \in H^{n+3}$. Similarly, by differentiating the last equation of (4.1) with respect to time, we can express $(1 - P(\underline{y}))\dot{\underline{w}} \in H^n$ as a smooth function of $\underline{u}$, $\underline{v}$, $\underline{w} \in H^n$, $\underline{y} \in H^{n+1}$. Thus we can insert or omit terms involving $(1 - P(\underline{y}))\underline{v}$, $(1 - P(\underline{y}))\dot{\underline{w}}$ and compensate for this by other terms involving only lower order derivatives.

We use this to write (4.1) in the following form

$$\begin{aligned} &\dot{\underline{v}} = \underline{w} \\ &\rho\dot{\underline{w}} = \frac{1}{2} P(\underline{y})\underline{\underline{A}}\, P(\underline{y})\underline{v} + (1 - P(\underline{y}))\Delta \\ &\qquad (1 - P(\underline{y}))\underline{v} + \tilde{\underline{\Phi}}\ \{\underline{y},\underline{u},\underline{v},\underline{w},p.q\} \\ &\qquad\qquad - P(\underline{y})\ddot{\underline{q}} \ . \end{aligned} \tag{4.2}$$

Here $\underline{\underline{A}}$ stands for

$$(\underline{\underline{A}}\ \underline{v})^i = \frac{1}{2} \int_{-\infty}^t a(t-\tau)A_{ij}^{pr} \frac{\partial}{\partial \zeta^s} \{\frac{\partial v^j}{\partial \zeta^q} \frac{\partial \zeta^q}{\partial y^r(\tau)} \frac{\partial \zeta^s}{\partial y^p(\tau)}\} d\tau$$

and Δ denotes the Laplace operator. The term $(1 - P(\underline{y}))\ \Delta(1 - P(\underline{y}))\underline{v}$ will be convenient later. Of course, another term compensating for it is contained in $\tilde{\underline{\Phi}}$.

We rewrite (4.2) as an evolution problem on a history space by introducing the "hat variables" as in chapter 3: $\hat{\underline{v}}(\sigma) = \underline{v}(\sigma t)$, $\hat{\underline{w}}(\sigma) = \underline{w}(\sigma t)$. The operator $\underline{\underline{A}}$ is split into two parts:

$$(\underline{\underline{A}}_1(t,\hat{\underline{y}},\sigma)\hat{\underline{v}}(\sigma))^i = \frac{1}{2}\ t \int_{-\infty}^0 a(\sigma t-\tau)A_{ij}^{pr}(\hat{\underline{y}}(\sigma),\underline{y}_0(\tau))$$

$$\frac{\partial}{\partial \zeta^s} [\frac{\partial \hat{v}^j(\sigma)}{\partial \zeta^q} \frac{\partial \zeta^q}{\partial y_0^r(\tau)} \frac{\partial \zeta^s}{\partial y_0^p(\tau)}] d\tau$$

$$(\underline{\underline{A}}_2(t,\hat{\underline{y}},\sigma)\hat{\underline{v}}(\sigma))^i = \frac{1}{2}\, t \int_0^\sigma a(\sigma t-\sigma' t) A_{ij}^{pr}(\hat{\underline{y}}(\sigma),\hat{\underline{y}}(\sigma'))$$

$$\frac{\partial}{\partial\zeta^s}\left[\frac{\partial\hat{v}^j(\sigma)}{\partial\zeta^q}\,\frac{\partial\zeta^q}{\partial\hat{y}^r(\sigma')}\,\frac{\partial\zeta^s}{\partial\hat{y}^p(\sigma')}\right]d\sigma' \quad .$$

Equation (4.2) can now be written in the following form:

$$\begin{aligned}
&\dot{\hat{\underline{v}}}(\sigma) = \sigma\hat{\underline{w}}(\sigma) \\
(4.3)\qquad &\dot{\hat{\underline{w}}}(\sigma) = \sigma \cdot \frac{1}{\rho}\,\{P(\hat{\underline{y}}(\sigma))(\underline{\underline{A}}_1 + \underline{\underline{A}}_2)P(\hat{\underline{y}}(\sigma))\hat{\underline{v}}(\sigma) \\
&\qquad + (1 - P(\hat{\underline{y}}(\sigma))\Delta(1 - P(\hat{\underline{y}}(\sigma))\hat{\underline{v}}(\sigma) + \\
&\qquad \hat{\underline{\Phi}}(\hat{\underline{v}},\hat{\underline{w}})(\sigma) + P(\hat{\underline{y}}(\sigma))\hat{\underline{g}}(\sigma)\} \quad .
\end{aligned}$$

It will turn out that (4.3) can be put into the context of an abstract result due to Hughes, Kato and Marsden [8]. In the following, we quote their assumptions and their result.

We are concerned with an evolution problem written in the form

$$(4.4)\qquad \dot{u} = -A(t,u)u + f(t,u),\ 0 \le t \le T,\ u(0) = \phi \quad ,$$

where u lies in a Banach space. Since the operator A is in applications a differential operator, it does not take any Banach space into itself, and we have to consider more than one topology. We shall therefore be dealing with three Banach spaces: $Y \subset X \subset Z$, all reflexive and separable with continuous and dense inclusions. (Hughes, Kato and Marsden introduce a fourth space Z', which for our application can be taken equal to X.)

On the space Z, a variable norm is considered. Let $N(Z)$ be the set of all norms in Z equivalent to the given one $\|\ \|_Z$. A metric on $N(Z)$ is given by

$$d(\|\ \|_\mu,\|\ \|_\nu) = \ln\max\{\sup_{0\neq z\in Z}\|z\|_\mu/\|z\|_\nu,\ \sup_{0\neq z\in Z}\|z\|_\nu/\|z\|_\mu\} \quad .$$

We now state the assumptions of Hughes, Kato and Marsden. All assumptions are supposed to hold for $t,t',\ldots\in[0,T]$ for some $T > 0$ and $w,w',\ldots\in W$, where W is an open set in Y. $\beta,\lambda_N,\mu_N,\ldots$ are arbitrary constants. The assumptions are:

(N) For $(t,w) \in [0,T] \times W$, there is $N(t,w) \in N(Z)$ such that

$$d(N(t,w),\ \|\ \|_Z) \le \lambda_N$$

$$d(N(t',w'),\ N(t,w)) \le \mu_N(|t-t'| + \|w-w'\|_X).$$

(S) There is an isomorphism $S(t,w) : Y \to Z$ such that

$\|S(t,w)\|_{Y,Z} \leq \lambda_s$, $\|S^{-1}(t,w)\|_{Z,Y} \leq \lambda_s^1$

$\|S(t',w') - S(t,w)\|_{Y,Z} \leq \mu_s(|t'-t| + \|w'-w\|_X)$.

(A1) $A(t,w) \in G(Z_{N(t,w)}, 1, \beta)$. This means that $-A(t,w)$ is a C_0-generator in Z (equipped with the norm $N(t,w)$) such that

$\|e^{-\tau A(t,w)}z\|_{N(t,w)} \leq e^{\beta\tau}\|z\|_{N(t,w)}$.

(A2) $S(t,w)A(t,w)S^{-1}(t,w) = A(t,w) + B(t,w)$, where

$B(t,w) \in B(Z)$, $\|B(t,w)\|_{Z,Z} \leq \lambda_B$.

(A3) $A(t,w) \in B(Y,X)$ with $\|A(t,w)\|_{Y,X} \leq \lambda_A$

and $\|A(t,w') - A(t,w)\|_{Y,X} \leq \mu_A\|w-w'\|_X$.

The mapping $t \mapsto A(t,w) \in B(Y,Z)$ is continuous in norm.

(A4) There is an element $y_0 \in W$ such that

$A(t,w)y_0 \in Y$, $\|A(t,w)y_0\| \leq \lambda_0$.

(f1) $f(t,w) \in Y$, $\|f(t,w)\|_Y \leq \lambda_f$, $\|f(t,w') - f(t,w)\|_X \leq \mu_f\|w'-w\|_X$ and the mapping $t \mapsto f(t,w) \in Z$ is continuous.

Under these assumptions, the following theorem is proved.

Theorem 4.1:

Assume (N), (S), (A1) - (A4) and (f1) hold. Then there is $\rho' > 0$ and $0 < T' \leq T$ such that for $\phi \in Y$ with $\|\phi-y_0\|_Y \leq \rho'$, equation (4.4) has a unique solution u on $[0,T']$ such that $u \in C([0,T'];W) \cap C^1([0,T'];X)$. Here ρ' depends on λ_s, λ_s', λ_N and $R := \mathrm{dist}(y_0,Y\setminus W)$, T' depends on all the constants occurring in the assumptions. When ϕ varies in Y subject to the restriction $\|\phi-y_0\| \leq \rho'$, the mapping $\phi \mapsto u(t)$ is Lipschitz continuous in the X-norm, uniformly for $t \in [0,T']$.

We now show how this applied to (4.3). For this, we have to identify the space Y, X and Z and the functions A, f, S and N.

The definition of the spaces requires that we specify a topology both for the spatial dependence and for the σ-dependence. Both have to be reflexive. We denote by $W^{0,n}$ the space of all functions $\hat{\underline{v}}(\sigma) : \sigma \in [0,1] \to H^n(\mathbb{R}^3,\mathbb{R}^3)$, which are square integrable over $[0,1]$ (in the Bochner sense). $W^{1,n}$ denotes the space of all functions

$[0,1] \to H^n(\mathbb{R}^3,\mathbb{R}^3)$, which have a square integrable derivative (with respect to σ). Moreover, we let $V^n = (W^{1,n+1} \cap W^{0,n+4}) \times (W^{1,n} \cap W^{0,n+3})$. Our choice of spaces is $Z = V^1$, $X = V^n$, $Y = V^{n+1}$, where $n > 4$. The operator $-A$ is identified with the underlined terms on the right hand side of (4.3), while f consists of the remaining terms. We take S to be equal to $(\frac{A}{\sigma} + \lambda)^n$, where λ is such that this operator is invertible. (As in chapter 3, it follows from the theory of elliptic equations that such a λ exists.)

This last choice trivially satisfies (A2) with $B = 0$. Moreover, our choice of topologies is such that all boundedness and Lipschitz conditions are satisfied, (under appropriate smoothness assumptions on the kernel a, the strain energy W and the data of the problem) i.e. the conditions (S), (A3) and (f1). As in most applications, there is a dense set of y_0 satisfying (A4) (any $y_0 \in V^{n+2}$ does).

It thus remains to identify a norm N such that (A1) holds. For this, we write

$$-A = \sigma\begin{pmatrix} 0 & 1 \\ \tilde{A} & 0 \end{pmatrix} .$$

The operator $\tilde{A}$ has a similar form as in [8], and the same argument as there shows that (A1) holds in the space $W^{0,1} \times W^{0,0}$, if this space is equipped with the norm

$$\|(\hat{\underline{v}},\hat{\underline{w}})\|^2_{N^*} = (\hat{v}, (-\tilde{A} + \lambda_0)\hat{v})_{L^2} + (\hat{w},\hat{w})_{L^2} .$$

Here $\lambda_0 > 0$ is chosen large enough that $-\tilde{A} + \lambda_0$ is invertible. We have to verify that (A1) also holds in higher Sobolev spaces. As far as the space dependence is concerned, this is easily achieved by using norms involving the operator S (see [10-12]). However, the topology of Z also involves a σ-derivative. Note that

$$\frac{d}{d\sigma} A(\hat{\underline{v}},\hat{\underline{w}}) = A\left(\frac{d\hat{\underline{v}}}{d\sigma}, \frac{d\hat{\underline{w}}}{d\sigma}\right) + \frac{dA}{d\sigma} (\hat{\underline{v}},\hat{\underline{w}}) .$$

Since $\frac{dA}{d\sigma}$ is bounded from Z into $W^{0,2} \times W^{0,1}$, we can satisfy (A1) by choosing the same norm for the σ-derivative. The perturbation given by $\frac{dA}{d\sigma}$ will not disturb the resolvent estimate required for the C_0-semigroup property.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2530	2. GOVT ACCESSION NO. AD-A130 503	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Local Existence and Uniqueness Theorem for a K-BKZ-Fluid		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Michael Renardy		8. CONTRACT OR GRANT NUMBER(s) MCS-7927062, Mod. 2. DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		12. REPORT DATE June 1983
		13. NUMBER OF PAGES 15
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Quasilinear Hyperbolic Equations, Viscoelastic Liquids, Semigroup theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We consider the three-dimensional motion of a viscolastic liquid occupying all space. The constitutive law is assumed to be of the form suggested by Kaye [13] and Bernstein, Kearsley and Zapas [2]. An existence and uniqueness result for solutions of the initial value problem on sufficiently short time intervals is proved using Kato's theory of quasilinear hyperbolic equations.

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